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On linear and exponential type non-linear lattices

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Abstract. A method of linearisation of the infinite Toda lattice is proposed, which enables one to recognise the relation between the infinite Toda lattice and the harmonic lattice straightforwardly.

1. Introduction

Since the integrable character of an exponential type non-linear lattice, the so-called Toda lattice [1], was recognised [2-4], it has received interest as a typical non-linear integrable model. As far as the finite (periodic) system is concerned, the integrability implies that the equations of motion of this non-linear lattice are canonically equivalent to those of the harmonic lattice. As for the infinite system, the situation seems to be rather different. In this case, although the inverse scattering method affords us a non-linear transformation such that the transformed variables (the scattering data) evolve linearly in time, it seems difficult to prove that these linearised equations are equivalent to the equations of motion of the harmonic lattice†.

It is therefore desirable if one can find a method of linearisation which enables one to recognise the relation between the infinite Toda lattice and the harmonic lattice clearly.

This paper is written with the purpose of proposing such a method. According to this method, it can be shown straightforwardly that the equations of motion of the infinite Toda lattice are reduced to those of the harmonic lattice, i.e. the former are governed by the latter, but not vice versa. So, within the framework of our method, they are not equivalent to each other‡.

2. Some preliminaries

Lattice vibrations of solids in one dimension are modelled by the Hamiltonian

$$\mathscr{H} = \sum_{-\infty<k<\infty} p_k^2/2m + \sum_{-\infty<k<\infty} \phi(x_{k+1}-x_k)$$

† The inverse scattering method also allows one to write down the dynamics of the Toda lattice in terms of normal mode Hamiltonian [5] which consists of non-soliton (continuous) and soliton (discrete) parts, such that the non-soliton part is equivalent to the normal mode Hamiltonian of the harmonic lattice.

‡ This conclusion, of course, relies on the particular method of linearisation presented here and does not exclude the possibility of proving the equivalence by other methods.

where m is the mass of atoms constituting the lattice, x_k denotes the displacement of the kth atom from its equilibrium position, p_k is the conjugate momentum and $\phi(r)$ stands for the interaction potential between the neighbouring atoms.

Equations of motion for this system are given by

$$\dot{x}_k = p_k/m \tag{2.1a}$$

$$\dot{p}_k = \phi'(x_{k+1}-x_k) - \phi'(x_k - x_{k-1}) \qquad \phi'(r) \equiv \mathrm{d}\phi(r)/\mathrm{d}r. \tag{2.1b}$$

For the harmonic lattice, the interaction potential $\phi(r)$ is specified by

$$\phi(r) = (\kappa/2)r^2 \qquad \kappa > 0$$

while in the Toda lattice $\phi(r)$ is given by [1]

$$\phi(r) = (a/b)\exp(-br) + ar \qquad ab > 0$$

so that (2.1b) turns out to be

$$\dot{p}_k = \kappa(x_{k+1} - 2x_k - x_{k-1}) \tag{2.2}$$

and

$$\dot{p}_k = a\{\exp[b(x_{k-1}-x_k)] - \exp[b(x_k - x_{k+1})]\} \tag{2.3}$$

respectively.

Without loss of generality, we can set in these equations all the constants m, κ, a and b equal to 1. Then it follows from (2.1a) and (2.2) that

$$\dot{\gamma}_k = \gamma_{k+1} - \gamma_{k-1} \tag{2.4}$$

with

$$\gamma_{2k-1} = -p_k \qquad \gamma_{2k} = x_k - x_{k+1}. \tag{2.5}$$

Similarly, from (2.1a) and (2.3),

$$\dot{A}_k = A_k(b_{k+1} - b_k) \qquad \dot{b}_k = A_k - A_{k-1} \tag{2.6}$$

with

$$A_k = \exp(x_k - x_{k+1}) \qquad b_k = -p_k.$$

Then our purpose is to show that (2.6) are reduced to (2.4).

3. Derivation of the results

In order to do this, we first consider the system of non-linear differential equations

$$\dot{y}_k = y_k(y_{k+1} - y_{k-1}) \qquad k = 1, 2, \ldots; \quad y_k(\neq 0) \in \mathbb{R}, \quad y_0 = 0 \tag{3.1}$$

which, by virtue of the results of a previous paper [6], can be reduced to†

$$\dot{\delta}_k = \delta_{k+1} \qquad k = 1, 2, \ldots \tag{3.2}$$

where no restrictions are imposed on the behaviour of the variables y_k at infinity ($k \to \infty$) except that $y_k \neq 0$.

† For the sake of simplicity, we have omitted the factor 4 in $\dot{\delta}_k = 4\delta_{k+1}$.

The reason for considering (3.1) instead of directly dealing with (2.6) is that one can relate easily the two semi-infinite ($k=1,2,\ldots$) systems with a doubly infinite ($k=0,\pm1,\pm2,\ldots$) one.

Thus we write a copy of (3.1)

$$\dot{\tilde{y}}_k=\tilde{y}_k(\tilde{y}_{k+1}-\tilde{y}_{k-1}) \qquad k=1,2,\ldots; \quad \tilde{y}_k(\neq 0)\in\mathbb{R}, \quad \tilde{y}_0=0 \tag{3.3}$$

which is reduced to

$$\dot{\tilde{\delta}}_k=\tilde{\delta}_{k+1} \qquad k=1,2,\ldots \tag{3.4}$$

and consider a doubly infinite system

$$\dot{\beta}_k=\beta_k(\beta_{k+1}-\beta_{k-1}) \qquad k=0,\pm1,\pm2,\ldots; \quad \beta_k(\neq 0)\in\mathbb{R}. \tag{3.5}$$

The differential equation for $k=0$

$$\dot{\beta}_0=\beta_0(\beta_1-\beta_{-1})$$

is rewritten as

$$\dot{y}_1=y_1y_2$$

with

$$y_1=\beta_0 \tag{3.6a}$$

$$y_2=\beta_1-\beta_{-1}. \tag{3.6b}$$

Similarly, the equation for $k=-1$

$$\dot{\beta}_{-1}=\beta_{-1}(\beta_0-\beta_{-2})$$

is the same as

$$\dot{\tilde{y}}_1=\tilde{y}_1\tilde{y}_2$$

with

$$\tilde{y}_1=\beta_{-1} \tag{3.7a}$$

$$\tilde{y}_2=\beta_0-\beta_{-2}. \tag{3.7b}$$

It follows from these that the system (3.5) for $k=-1,0$ is equivalent to the composite system of (3.1) and (3.3) for $k=1$, where in order to make y_k, $\tilde{y}_k$ ($k=1,2$) and β_k ($k=-2,-1,0,1$) different from zero, the following restrictions are imposed:

$$\beta_k\neq 0 \qquad k=-2,-1,0,1 \qquad \beta_1-\beta_{-1}\neq 0 \qquad \beta_0-\beta_{-2}\neq 0 \tag{3.8a}$$

$$y_1,\tilde{y}_1\neq 0 \qquad \tilde{y}_1+y_2\neq 0 \qquad y_1-\tilde{y}_2\neq 0. \tag{3.8b}$$

It further follows from (3.6b) and (3.5) that

$$\dot{y}_2=[\beta_1(\beta_2-\beta_0)-\beta_{-1}(\beta_0-\beta_{-2})]\equiv y_2(y_3-y_1)$$

which defines y_3 in terms of $\{\beta_k\}_{k=-2}^{k=1}\equiv\{\beta_k\}_{-2}^{1}$, or β_2 in terms of $(\{y_k\}_1^3,\{\tilde{y}_k\}_1^2)$.

It is obvious that for each $\{\beta_k\}_{-2}^{1}$ satisfying the conditions (3.8a), there exists a single value of β_2, denoted by $\beta_2^{\dagger}$, that makes y_3 equal to zero, and also for each $(\{y_k\}_1^2,\{\tilde{y}_k\}_1^2)$ satisfying (3.8b), there exists a value of y_3, denoted by $y_3^{\dagger}$, that makes β_2 equal to zero.

Similarly, taking account of (3.7b) and (3.5), we have

$$\dot{\tilde{y}}_2=[\beta_0(\beta_1-\beta_{-1})-\beta_{-2}(\beta_{-1}-\beta_{-3})]\equiv\tilde{y}_2(\tilde{y}_3-\tilde{y}_1)$$

which defines $\tilde{y}_3$ in terms of $\{\beta_k\}_{-3}^{1}$, or β_{-3} in terms of $(\{y_k\}_1^2, \{\tilde{y}_k\}_1^3)$. Again, for each $\{\beta_k\}_{-2}^{1}$ there exists a value of β_{-3}, denoted by $\beta_{-3}^{\dagger}$, that makes $\tilde{y}_3$ equal to zero, and for each $(\{y_k\}_1^2, \{\tilde{y}_k\}_1^2)$ there exists a value of $\tilde{y}_3$, denoted by $\tilde{y}_3^{\dagger}$, that makes β_{-3} equal to zero.

In addition to the restrictions (3.8), we must impose the following so as to make y_k, $\tilde{y}_k$ $(k=1,2,3)$ and β_k $(k=-3,\ldots,2)$ different from zero†:

$$\beta_2 \neq 0, \beta_2^{\dagger} \qquad y_3 \neq 0, y_3^{\dagger} \tag{3.9}$$

$$\beta_{-3} \neq 0, \beta_{-3}^{\dagger} \qquad \tilde{y}_3 \neq 0, \tilde{y}_3^{\dagger}. \tag{3.10}$$

Under the restrictions (3.8) and (3.9), the system (3.5) for $k=-1,0,1$ is equivalent to the composite system of (3.1) for $k=1,2$ and (3.3) for $k=1$. Similarly, under the restrictions (3.8), (3.9) and (3.10), the system (3.5) for $k=-2,-1,0,1$ is equivalent to the composite system of (3.1) and (3.3), both for $k=1,2$.

We can continue this procedure to recognise that, under certain restrictions on the variables, the doubly infinite system (3.5) is equivalent to the composite system of (3.1) and (3.3). These restrictions are solely imposed so as to make all the variables $\{y_k\}_1^{\infty}$, $\{\tilde{y}_k\}_1^{\infty}$ and $\{\beta_k\}_{-\infty}^{\infty}$ different from zero, which is necessary for the correspondence between $(\{y_k\}_1^{\infty}, \{\tilde{y}_k\}_1^{\infty})$ and $\{\beta_k\}_{-\infty}^{\infty}$ to be defined in the manner just described above.

Since the systems (3.1) and (3.3) are reduced to (3.2) and (3.4), respectively, this result immediately implies that, under these restrictions, the system (3.5) is reduced to the composite system of (3.2) and (3.4).

Now, it is easy to see that this composite system is equivalent to‡

$$\dot{z}_k = z_{k+1} - z_{k-1} \qquad \dot{\tilde{z}}_k = \tilde{z}_{k+1} - \tilde{z}_{k-1} \qquad k=1,2,\ldots \qquad z_0 = \tilde{z}_0 = 0$$

which is further equivalent to (2.4) with the transformation given by

$$\begin{aligned} &z_1 = \gamma_0 \qquad \tilde{z}_1 = \gamma_{-1} \qquad z_2 = \gamma_2 - \gamma_{-1} \qquad \tilde{z}_2 = \gamma_0 - \gamma_{-2} \\ &z_3 = \gamma_2 - \gamma_0 + \gamma_{-2} \qquad \tilde{z}_3 = \gamma_1 - \gamma_{-1} + \gamma_{-3} \\ &z_4 = \gamma_3 - \gamma_1 + \gamma_{-1} - \gamma_{-3} \qquad \tilde{z}_4 = \gamma_2 - \gamma_0 + \gamma_{-2} - \gamma_{-4}. \end{aligned}$$

etc. On the other hand, (3.5) are transformed to (2.6) by

$$A_k = \beta_{2k}\beta_{2k-1} \qquad b_k = \beta_{2k-1} + \beta_{2k-2} \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{3.11}$$

We have thus established the existence of a mapping from $\{\gamma_k\}_{-\infty}^{\infty}$ to $(\{A_k\}_{-\infty}^{\infty}, \{b_k\}_{-\infty}^{\infty})$ such that the time change of $\{\gamma_k\}_{-\infty}^{\infty}$ according to (2.4) causes the evolution (2.6) of $(\{A_k\}_{-\infty}^{\infty}, \{b_k\}_{-\infty}^{\infty})$.

Besides the fact that our result does not claim the equivalence of the infinite Toda lattice to the harmonic lattice§, it should be noticed that, although we have got rid of the restriction that the motion of the lattice rapidly decreases at infinities‖, which is inevitably imposed in the inverse scattering treatment, we have encountered another kind of restriction. In order to avoid overcomplication, we do not give any detailed

† It should be noticed that these conditions actually concern the initial values of the infinite systems (3.1), (3.3) and (3.5), since if they are satisfied initially it remains so at any time.

‡ The transformation leading to this equivalence is given by $\delta_i = z_i$, $\tilde{\delta}_i = \tilde{z}_i$ $(i=1,2)$, $\delta_3 = z_3 - z_1$, $\tilde{\delta}_3 = \tilde{z}_3 - \tilde{z}_1$, $\delta_4 = z_4 - 2z_2$, $\tilde{\delta}_4 = \tilde{z}_4 - 2\tilde{z}_2, \ldots$.

§ This stems from the fact that the mapping $\nu_k = \delta_{k+1}/\delta_1$ (see the expression below (14) of [6]) is not one to one. The same situation also arises in (3.11).

‖ This is because our method allows one to attack the problem from some finite point, say $k=0$, while in the inverse scattering method one attacks from the infinities $(k=\pm\infty)$.

account of them, but only remark that on account of these restrictions it becomes rather ambiguous whether or not our method applies to the case of periodic boundary conditions.

We also remark that all the non-linear transformations involved in our method, i.e. the transformations between $(\{y_k\}_1^\infty, \{\tilde{y}_k\}_1^\infty)$ and $(\{\nu_k\}_1^\infty, \{\tilde{\nu}_k\}_1^\infty)$, between $(\{\nu_k\}_1^\infty, \{\tilde{\nu}_k\}_1^\infty)$ and $(\{\delta_k\}_1^\infty, \{\tilde{\delta}_k\}_1^\infty)$ (see [6]), between $(\{y_k\}_1^\infty, \{\tilde{y}_k\}_1^\infty)$ and $\{\beta_k\}_{-\infty}^\infty$, and the one between $\{\beta_k\}_{-\infty}^\infty$ and $(\{A_k\}_{-\infty}^\infty, \{b_k\}_{-\infty}^\infty)$, admit some kinds of concrete expressions. The situation should be compared with the method of inverse scattering [4], where the non-linear transformation used to linearise (2.6) is afforded by solving the scattering problem associated with a certain infinite Jacobi matrix†, and in order to obtain the expression of $(\{A_k\}_{-\infty}^\infty, \{b_k\}_{-\infty}^\infty)$ in terms of the scattering data, one must solve the Marcenko equation.

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† This procedure apparently corresponds to the transformation from $(\{y_k\}_1^\infty, \{\tilde{y}_k\}_1^\infty)$ to $(\{\nu_k\}_1^\infty, \{\tilde{\nu}_k\}_1^\infty)$, or to $(\{\delta_k\}_1^\infty, \{\tilde{\delta}_k\}_1^\infty)$ in our method.